Practices before the class (April 12)

- (T/F) If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$ and if $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z}$ must be in $W^\perp$.
- (T/F) For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W.
- (T/F) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.
- (T/F) The best approximation to y by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$.

Practices before the class (April 12)

- (T/F) If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$ and if $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z}$ must be in $W^\perp$. True. Recall from Section 6.1 that $W^\perp$ denotes the set of all vectors orthogonal to a subspace W.
- (T/F) For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W\, \mathbf{y}$ is orthogonal to W. True by the Orthogonal Decomposition Theorem.
- (T/F) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W. True. The orthogonal decomposition in Theorem 8 is unique.
- (T/F) The best approximation to y by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W\, \mathbf{y}$. False. The Best Approximation Theorem says that the best approximation to $\mathbf{y}$ is $\text{proj}_W \mathbf{y}$.

6.4 The Gram-Schmidt Process

The Gram-Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any nonzero subspace of $\mathbb{R}^n$. We will use the next example to introduce the detail of the process.

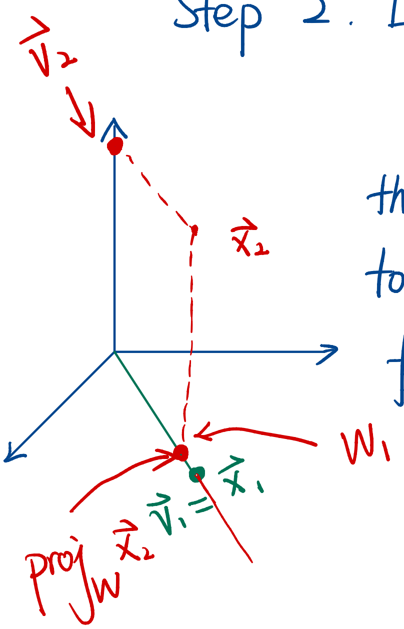
Example 1. Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus is a basis for a subspace W of $\mathbb{R}^4$. Construct an orthogonal basis for W.

ANS: Step 1. Let $\vec{v}_1 = \vec{x}_1$ and $W_1 = \text{span}\{\vec{x}_1\} = \text{span}\{\vec{v}_1\}$.

Step 2. Let

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1} \vec{x}_2$$

then $\vec{v}_2$ is the component of $\vec{x}_2$ orthogonal to $\vec{x}_1$ and $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for $W_2 = \text{span}\{\vec{v}_1, \vec{v}_2\}$.



Compute
$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1} \vec{x}_2$$
$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$
$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

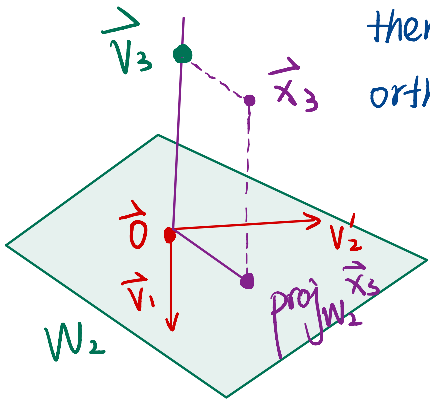
Step 2′ (optional). We can scale $\vec{v}_2$ to simplify the later computation. So we have

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2\,' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We update $W_2 = \text{span}\{\vec{v}_1, \vec{v}_2\,'\}$

Step 3. Let

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$$



then $\vec{v}_3$ is the component of $\vec{x}_3$ orthogonal to W_2 and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ an orthogonal set.

We compute

$$\text{proj}_{W_2} \vec{x}_3 = \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{x}_3 \cdot \vec{v}_2\,'}{\vec{v}_2\,' \cdot \vec{v}_2\,'} \vec{v}_2\,'$$

$$= \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Note $\vec{v}_3$ is in W since $\vec{x}_3$ and $\text{proj}_{W_2} \vec{x}_3$ are both in W_3.

Thus $\{\vec{v}_1, \vec{v}_2', \vec{v}_3\}$ is an orthogonal set of nonzero vectors so they are linearly independent.

Since W is 3-dim'l. $\{\vec{v}_1, \vec{v}_2', \vec{v}_3\}$ is an orthogonal basis for W by the Basis Theorem.

Theorem 11 The Gram-Schmidt Process

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \operatorname{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \leq k \leq p$$

Orthonormal Bases

- An orthonormal basis is constructed easily from an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$: simply normalize (i.e., "scale") all the $\mathbf{v}_k$.
- When working problems by hand, this is easier than normalizing each $\mathbf{v}_k$ as soon as it is found (because it avoids unnecessary writing of square roots).

Example 2. Find an orthonormal basis of the subspace spanned by the vectors in **Example 1.**

Recall from Example 1.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2{}' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

An orthonormal basis is

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{\vec{v}_2{}'}{\|\vec{v}_2{}'\|} = \cancel{\frac{1}{\sqrt{9+1+1+1}}}^{\frac{1}{2\sqrt{3}}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{3}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \cancel{\frac{1}{\sqrt{\frac{4+1+1}{9}}}}^{\frac{1}{\sqrt{6}/9} = \frac{3}{\sqrt{6}}}\begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

QR Factorization of Matrices

Theorem 12 The QR Factorization

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\operatorname{Col} A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Example 3. Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

ANS: First notice that the columns of A are $\vec{x}_1, \vec{x}_2, \vec{x}_3$ given in Example 1. We found the orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ in Example 2. So we have them as columns of Q:

$$Q = [\vec{u}_1 \ \ \vec{u}_2 \ \ u_3] = \begin{bmatrix} 1/2 & -3/2\sqrt{3} & 0 \\ 1/2 & 1/2\sqrt{3} & -2/\sqrt{6} \\ 1/2 & 1/2\sqrt{3} & 1/\sqrt{6} \\ 1/2 & 1/2\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

To find R, first notice $Q^TQ = I$. (Thm 6 in §6.2, since Q has orthonormal columns). So we have

$$Q^T A = Q^T(QR) = IR = R$$

We compute

$$R = Q^TA = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2\sqrt{3} & 1/2\sqrt{3} & 1/2\sqrt{3} & 1/2\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/2\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

Exercise 4. Find an orthogonal basis for the column space of the given matrix

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Solution. Call the columns of the matrix $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-2)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{3}{2}\mathbf{v}_1 - \left(-\frac{1}{2}\right)\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Thus an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Exercise 5. The columns of Q were obtained by applying the Gram-Schmidt process to the columns of A. Find an upper triangular matrix R such that $A = QR$.

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}, Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

Solution. Since A and Q are given, $R = Q^T A = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$